Engineering Notes

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Solution of the Lifting Line Equation for Twisted Elliptic Wings

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ALTHOUGH lifting surface calculations are now routinely carried out with great accuracy, the Prandtl lifting line theory remains a much cherished design tool for rapid estimation of the loading carried by classically shaped wings. Induction effects of nearby vortices (wing-tail interaction, formation flying) or the lift due to flap deflection, for example, are cases where lifting line calculations often lead to excellent engineering approximations. In many cases the circulation carried by a wing may be accurately estimated as equal to that on a wing of the same aspect ratio but of elliptic planform twisted so as to have the spanwise upwash distribution of the original wing. This Note provides an exact inversion of the lifting line equation for wings of elliptic planform with arbitrary spanwise upwash.

Suppose a wing of span $2s$ and (large) aspect ratio A flying at uniform speed U is twisted so that the angle of attack wU varies along the span. The circulation Γ is related to the upwash w through Prandtl's lifting line equation (e.g., Ref. 1)

$$\Gamma = \pi w - c/4 \int_{-s}^{s} d\Gamma/d\eta (y - \eta)^{-1} d\eta \qquad (1)$$

where c is the local chord. If, as usual, Γ is represented by the Fourier series

$$\Gamma = sU \sum_{n=0}^{\infty} a_n \sin(n + 1)\theta \qquad (2)$$

where $\theta = \cos^{-1}(y/s)$, the total lift and rolling moment coefficients are

$$C_L = \pi A a_0/4 \qquad (3)$$

$$C_M = \pi A a_1/4 \qquad (4)$$

The physical spanwise variable $\bar{y} = y/s$ may be conveniently reintroduced into Eq. (2) using the identity

$$\sin(n + 1)\theta = (1 - \bar{y}^2)^{1/2} U_n(\bar{y})$$

where U_n is the Chebyshev polynomial of the second kind.[2] Thus instead of Eq. (2) we may use the equivalent form

$$\Gamma(y) = sU(1 - \bar{y}^2)^{1/2} \sum_{n=0}^{\infty} a_n U_n(\bar{y}) \qquad (5)$$

Substitution of Eq. (5) into Eq. (1) and carrying out the operations in the left hand term leads to the algebraic relation

$$w(\bar{y}) = (2\pi)^{-1} U \sum_{0}^{\infty} a_n \times [2sc^{-1}(1 - \bar{y}^2)^{1/2} + (m + 1)\pi/2] U_n(y) \qquad (6)$$

as an alternate to the more usual one involving the angular variable θ.

But for a wing of elliptic planform

$$2sc^{-1} = \pi A/4(1 - \bar{y}^2)^{-1/2}$$

so that Eq. (6) simplifies to

$$w(\bar{y}) = U/8 \sum_{0}^{\infty} a_n[A + 2(n + 1)] U_n(\bar{y}) \qquad (7)$$

Using the orthogonality property of the Chebyshev polynomials Eq. (7) is readily inverted to give the formula for the Fourier coefficients

$$a_n = 16 (\pi U)^{-1}[A + 2(n + 1)]^{-1} \times \int_{-1}^{1} (1 - \bar{y}^2)^{1/2} U_n(\bar{y}) w(\bar{y}) d\bar{y} \qquad (8)$$

Equations (8) and (2) together form the explicit series solution for the circulation distribution on an elliptic wing with arbitrary spanwise upwash:

$$\Gamma(\bar{y}) = \pi^{-1} 16s \sum_{0}^{\infty} [A + 2(n + 1)]^{-1} \times \int_{-1}^{1} (1 - \bar{y}^2)^{1/2} U_n(\bar{y}) w(\bar{y}) d\bar{y} \qquad (9)$$

It may be easily shown that the lift and rolling moment coefficients calculated using Eqs. (3, 4, and 8) are identical to expressions obtainable by reverse flow methods.[3]

Example: sinusoidally twisted wing

The concept of a sinusoidally twisted lifting line was originally introduced by Prandtl himself.[4] Analogous to Prandtl's result for an infinitely long lifting line is the finite span result obtained by taking

$$w(\bar{y}) = \hat{w} e^{iks\bar{y}}$$

in Eq. (9):

$$\Gamma(\bar{y}) = 16(\pi k)^{-1} \hat{w} \sum_{0}^{\infty} (n + 1) \times [A + 2(n + 1)]^{-1} i^n J_{n+1}(ks) U_n(\bar{y}) \qquad (10)$$

where J_m is a Bessel function of the first kind.[2] Equation (10) which is valid for all wave numbers k may be used as a basis for constructing solutions by Fourier superposition.

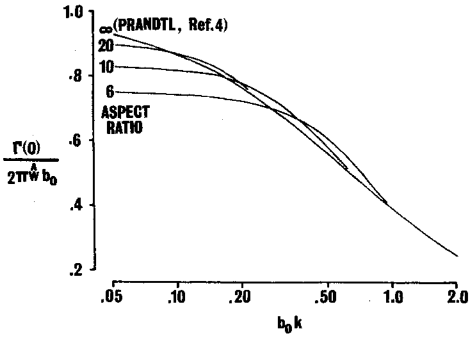

Fig. 1 Circulation around the midspan section of a sinusoidally twisted lifting line.

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Equation (10) may now be compared with Prandtl's result for the infinite lifting line.[4] On the wing centerline $\bar{y} = 0$, reduction of Eq. (10) gives

$$(2\pi\hat{w}b_0)^{-1}\Gamma(0) = A(A+2)^{-1}J_0(\pi A b_0 k/4) + A\sum_{m=0}^{\infty}(A/2+2m)[(A/2+2m)^2-1]^{-1} \times J_{2m}(\pi A b_0 k/4) \quad (11)$$

where b_0 is the semichord measured at midspan. The analogous result for the infinitely long lifting line is[4]

$$(2\pi\hat{w}b_0)^{-1}\,|\Gamma(0)|_{A=\infty} = (1+\pi b_0 k/2)^{-1} \quad (12)$$

Figure 1 shows how the centerline circulation calculated from Eq. (11) approaches the limit Eq. (12) for large aspect ratio.

References

[1] Ashley, H. and Landahl, M., *Aerodynamics of Wings and Bodies*, Addison-Wesley, Reading, Mass., 1965, Chap. 7, pp. 137–142.

[2] Abramowitz, M. and Stegun, I. A., eds., *Handbook of Mathematical Functions*, Dover, New York, 1965.

[3] Heaslet, M. A. and Spreiter, J. R., "Reciprocity Relations in Aerodynamics," TN 2700, 1952, NACA.

[4] Prandtl, L. and Betz, A., "Vier Abhandlungen zur Hydrodynamik und Aerodynamik," *Im Selbstverlag des Kaiser Wilhelm-Instituts für Strömungsforschung,* Göttingen, 1927.

Turbulent Skin Friction for Tapered Wings

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Nomenclature

b	= wing span
$c(\xi)$	= wing chord at spanwise station
c_f	= average skin-friction coefficient
c_{f_0}	= average skin-friction coefficient for zero taper ratio
$c_f(\xi)$	= skin-friction coefficient at spanwise station
c_r	= wing root chord
c_t	= wing tip chord
R	= Reynolds number $[U_\infty c(\xi)/\nu]$
R_r	= Reynolds number based on root chord
U_∞	= freestream velocity
x	= spanwise station
ξ	= $x/(b/2)$
λ	= taper ratio (c_t/c_r)
ν	= kinematic viscosity
$\log_{10}(\ )$	= logarithm to base 10
$\ln(\ )$	= logarithm to base e

THE problems involved in a rapid determination of the skin-friction drag for tapered wings have been discussed by both E. J. Hopkins[1] and A. Barkhem.[2] Very simply, the main problem resolves to one of estimating the over-all skin friction using a Reynolds number based on some average chord, usually the wing mean aerodynamic chord, which purports to represent the entire wing. This method is rapid but may result in small but significant errors (e.g., 3–4% for highly tapered wings at low Reynolds numbers). The alternative is to perform a spanwise integration of the local skin-friction coefficient. This is accurate but becomes simple only if the skin-friction formula used is simple.

The method developed below is both simple to use and very accurate in its results. It is derived on the basis of Reynolds numbers referred to the wing root-chord that are of practical interest in aircraft design.

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From the Prandtl-Schlichting turbulent skin-friction formula for a smooth flat plate, the local skin friction is[3]

$$c_f(\xi) = 0.455/(\log_{10}R)^{2.58} \quad (1)$$

The general planform considered is shown in Fig. 1. The local chord can be expressed as a function of planform geometry as

$$c(\xi)/c_r = 1 - (1-\lambda)\xi \quad (2)$$

Considering the Reynolds number, we find

$$R = U_\infty c/\nu = (U_\infty c_r/\nu)[1-(1-\lambda)\xi] = R_r[1-(1-\lambda)\xi] \quad (3)$$

Substitution of Eq. (3) into Eq. (1) gives the local skin-friction coefficient as a function of the root-chord Reynolds number and planform geometry. Thus,

$$c_f(\xi) = 0.455/\{\log_{10}R_r[1-(1-\lambda)\xi]\}^{2.58} \quad (4)$$

We can define the average skin-friction coefficient/unit area of surface by

$$c_f = \left[\int_0^1 c_f(\xi)c(\xi)d\xi\right]\Big/\left[\int_0^1 c(\xi)d\xi\right] \quad (5)$$

One can then write

$$c_f = \left[\int_0^1 c_f(\xi)c(\xi)/c_r d\xi\right]\Big/\left[\int_0^1 c(\xi)/c_r d\xi\right] \quad (6)$$

which with Eqs. (2) and (4) gives

$$c_f = \frac{2}{1+\lambda}\,\frac{0.455}{[\log_{10}R_r]^{2.58}} \times \int_0^1 \frac{1-(1-\lambda)\xi d\xi}{\{1+\ln[1-(1-\lambda)\xi]/\ln R_r\}^{2.58}} \quad (7)$$

Expanding the denominator in an infinite series gives

$$c_f = \frac{2}{1+\lambda}\,\frac{0.455}{(\log_{10}R_r)}\,{}_{2.58}\int_0^1 [1-(1-\lambda)\xi] \times \{1 - 2.58\ln[1-(1-\lambda)\xi]/\ln R_r + 4.6182[\ln[1-(1-\lambda)\xi]/\ln R_r]^2 + \ldots\}d\xi \quad (8)$$

It is easily shown that this integrand series is convergent as long as

$$\lambda > 1/R_r \quad (9)$$

For all practical purposes, this puts no restriction on λ for moderate to large R_r. Thus, integrating Eq. (8) term by

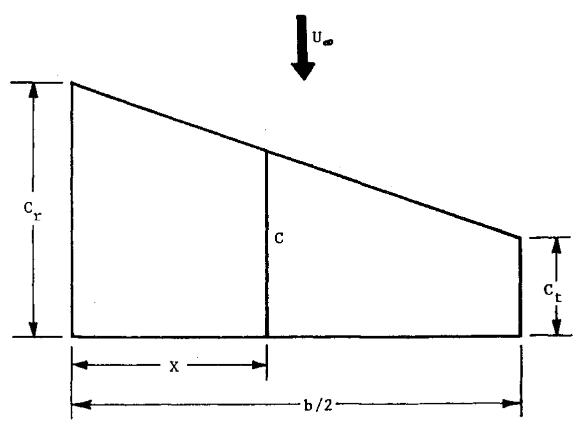

Fig. 1 Wing planform geometry.